Q1a

1a

For each of the following, find $\frac{dy}{dx}$ in terms of x:

(a) $y = -\frac{5}{4}x^3 + \frac{3}{5}x^2 - x\sqrt{2} + \pi$ [2]

(b) $y = \frac{3}{2}x^{\frac{4}{5}} - \frac{10}{3}x^{-\frac{4}{5}}$ [2]

a) $\frac{dy}{dx} = -\frac{5}{4}(3)x^2 + \frac{3}{5}(2)x^1 - \sqrt{2}$

$= \boxed{-\frac{15}{4}x^2 + \frac{6}{5}x - \sqrt{2}}$

Q1b

1b

For each of the following, find $\frac{dy}{dx}$ in terms of x:

(a) $y = -\frac{5}{4}x^3 + \frac{3}{5}x^2 - x\sqrt{2} + \pi$ [2]

(b) $y = \frac{3}{2}x^{\frac{4}{5}} - \frac{10}{3}x^{-\frac{4}{5}}$ [2]

b) $\frac{dy}{dx} = \frac{3}{2}\left(\frac{4}{5}\right)x^{\frac{4}{5}-1} - \frac{10}{3}\left(-\frac{4}{5}\right)x^{-\frac{4}{5}-1}$

$= \boxed{\frac{6}{5}x^{-\frac{1}{5}} + \frac{8}{3}x^{-\frac{9}{5}}}$

Q2

2

Given that $y = \left(\frac{1}{x} - \frac{1}{x\sqrt{x}}\right)^2$, $x > 0$, find $\frac{dy}{dx}$.

[4]

$$y = \left(\frac{1}{x} - \frac{1}{x\sqrt{x}}\right)\left(\frac{1}{x} - \frac{1}{x\sqrt{x}}\right)$$

$$= \left(\frac{1}{x}\right)^2 - 2\left(\frac{1}{x\sqrt{x}}\right)\left(\frac{1}{x}\right) + \left(\frac{1}{x\sqrt{x}}\right)^2$$

$$= \frac{1}{x^2} - \frac{2}{x^{5/2}} + \frac{1}{x^3}$$

$$= x^{-2} - 2x^{-5/2} + x^{-3}$$

$$\frac{dy}{dx} = -2x^{-3} - 2\left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1} + (-3)x^{-4}$$

$$= \boxed{-2x^{-3} + 5x^{-7/2} - 3x^{-4}}$$

Q3a

3a

For each of the following, find $\frac{dy}{dx}$ in terms of x:

(a) $y = \frac{2x^3 - 5x^2 - 3x}{2x + 1}$

[3]

(b) $y = \left(\sqrt{x} + 3 - \frac{4}{\sqrt{x}}\right)^2$

[4]

a) Factorise the numerator

$$2x^3 - 5x^2 - 3x = x(2x^2 - 5x - 3)$$

$$= x(x-3)(2x+1)$$

$$\therefore y = \frac{x(x-3)\cancel{(2x+1)}}{\cancel{2x+1}} = x(x-3) = x^2 - 3x$$

$$\boxed{\frac{dy}{dx} = 2x - 3}$$

Q3b

3b

For each of the following, find $\frac{dy}{dx}$ in terms of x:

(a) $y = \frac{2x^3 - 5x^2 - 3x}{2x + 1}$ [3]

(b) $y = \left(\sqrt{x} + 3 - \frac{4}{\sqrt{x}}\right)^2$ [4]

b) $y = \left(\sqrt{x} + 3 - \frac{4}{\sqrt{x}}\right)\left(\sqrt{x} + 3 - \frac{4}{\sqrt{x}}\right)$

$= x + 6\sqrt{x} - \frac{24}{\sqrt{x}} + \frac{16}{x} + 1$

$= x + 6x^{\frac{1}{2}} - 24x^{-\frac{1}{2}} + 16x^{-1} + 1$

$$\boxed{\frac{dy}{dx} = 1 + 3x^{-\frac{1}{2}} + 12x^{-\frac{3}{2}} - 16x^{-2}}$$

Q4a

4a

For each of the following, use the chain rule to find $\frac{dy}{dx}$ in terms of x:

(a) $y = \left(\frac{1}{\sqrt{x}} + 3x^2\right)^3$ [4]

$u = x^{-\frac{1}{2}} + 3x^2$

(b) $y = \sqrt{\frac{1}{\sqrt{x} + \frac{1}{\sqrt{x}}}}$ [4]

a) $u = x^{-\frac{1}{2}} + 3x^2$ $\quad y = u^3$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} + 6x$ $\quad \frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = \left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x\right)\left(3u^2\right)$

sub $u = x^{-\frac{1}{2}} + 3x^2$

$\frac{dy}{dx} = \left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x\right)3\left(x^{-\frac{1}{2}} + 3x^2\right)^2$

$$\boxed{\frac{dy}{dx} = 3\left(6x - \frac{1}{2x\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}} + 3x^2\right)^2}$$

Q4b

4b

For each of the following, use the chain rule to find $\frac{dy}{dx}$ in terms of x:

(a) $y = \left(\frac{1}{\sqrt{x}} + 3x^2\right)^3$ [4]

(b) $y = \sqrt{\frac{1}{\sqrt{x}+\frac{1}{\sqrt{x}}}} = \left(\underbrace{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}_{u}\right)^{-\frac{1}{2}}$ [4]

b) $u = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ $\quad y = u^{-\frac{1}{2}}$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$ $\quad \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2}\left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right)\left(-u^{-\frac{3}{2}}\right)$$

$$= \frac{1}{4}\left(x^{-\frac{3}{2}} - x^{-\frac{1}{2}}\right)\left(u^{-\frac{3}{2}}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{4}\left(x^{-\frac{3}{2}} - x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^{-\frac{3}{2}}}$$

You can simplify this further...

$$\frac{dy}{dx} = \frac{1 - x}{4x(x+1)}\sqrt{\frac{\sqrt{x}}{x+1}}$$

Q5

5

A curve has the equation $y = x\sqrt{x} + \frac{48}{\sqrt{x}}$, $x > 0$. Find the coordinates of the point on the curve where the gradient is 0. [5]

$$y = x^{\frac{3}{2}} + 48x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 48\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = 0$$

$$\frac{3}{2}x^{\frac{1}{2}} - 24x^{-\frac{3}{2}} = 0$$

$$\frac{3}{2}x^{-\frac{3}{2}}\left(x^2 - 16\right) = 0$$

$x = 4, \cancel{-4}$ ← reject since $x > 0$

Find corresponding y coordinate

$$y = 4\sqrt{4} + \frac{48}{\sqrt{4}} = 32$$

POINT: $\boxed{(4, 32)}$

Q6

6

The function f is defined by $f(x) = 2x^3 + px^2 + 3x - 16$. Determine the range of values for p for which the equation $f'(x) = 0$ has at least one real solution.

[5]

$f'(x) = 6x^2 + 2px + 3 = 0$

discriminant $\geqslant 0$ — for the quadratic to have at least one real solution

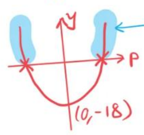
$b^2 - 4ac \geqslant 0$

$(2p)^2 - 4(6)(3) \geqslant 0$

$4p^2 - 72 \geqslant 0$

$p^2 - 18 \geqslant 0$

sketch $p^2 - 18 \geqslant y$



Find points of intersection with p-axis

$0 = p^2 - 18$

$18 = p^2$

$p = \pm\sqrt{18} = \pm 3\sqrt{2}$

$\therefore$ $\boxed{p \geqslant 3\sqrt{2} \quad p \leqslant -3\sqrt{2}}$

Q7a

7a

Given that $y = \left(\frac{1}{x} + x\right)^4$, $x \neq 0$

u

(a) use the chain rule to find $\frac{dy}{dx}$

[3]

(b) find the coordinates of any stationary points and determine their nature

[4]

(c) sketch the curve.

[3]

a) $u = x^{-1} + x$ $\qquad y = u^4$ $\qquad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{du}{dx} = -x^{-2} + 1$ $\qquad \frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = 4u^3(1 - x^{-2})$

sub in $u = x^{-1} + x$

$= 4(x^{-1} + x)^3(1 - x^{-2})$

$\boxed{\frac{dy}{dx} = 4\left(\frac{1}{x} + x\right)^3\left(1 - \frac{1}{x^2}\right)}$

Q7b

7b

Given that $y = \left(\frac{1}{x} + x\right)^4$, $x \neq 0$

(a) use the chain rule to find $\frac{dy}{dx}$ [3]

$\frac{dy}{dx} = 4\left(\frac{1}{x} + x\right)^3\left(1 - \frac{1}{x^2}\right) = 0$

(b) find the coordinates of any stationary points and determine their nature [4]

(c) sketch the curve. [3]

b) $\frac{dy}{dx} = 4\left(\frac{1}{x} + x\right)^3\left(1 - \frac{1}{x^2}\right) = 0$

$4\left(\frac{1+x^2}{x}\right)^3\left(\frac{x^2-1}{x^2}\right) = 0$ $\quad x \neq 0$

$1 + x^2 = 0$ $\qquad x^2 - 1 = 0$

$x^2 = -1$ $\qquad x^2 = 1$

no real solutions $\qquad x = \pm 1$

When $x = 1$, $y = \left(\frac{1}{1} + 1\right)^4 = 2^4 = 16$

$x = -1$, $y = \left(\frac{1}{-1} + -1\right)^4 = (-2)^4 = 16$

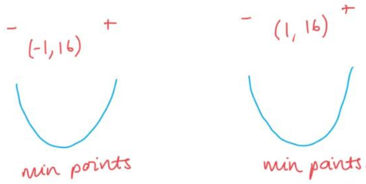
SPs: $(-1, 16)$ $\quad (1, 16)$

Determine the nature of the SPs by finding the sign of the gradient either side of the SPs.

$x \neq 0$, so there must be an asymptote at $x = 0$. Therefore we need to find the gradient either side of the asymptote.

x	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
$\frac{dy}{dx}$	-46.875	187.5	-187.5	46.875

asymptote at $x = 0$



min points at $(-1, 16)$ and $(1, 16)$

Q7c

7c

Given that $y = \left(\frac{1}{x} + x\right)^4$, $x \neq 0$

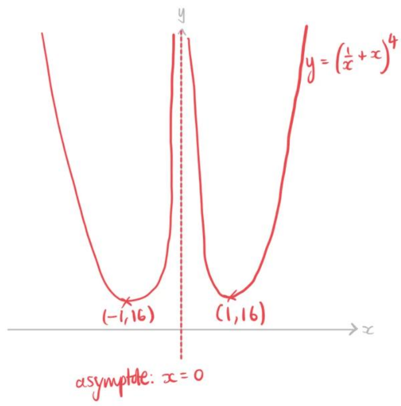
(a) use the chain rule to find $\frac{dy}{dx}$ [3]

(b) find the coordinates of any stationary points and determine their nature [4]

min points at $(-1, 16)$ and $(1, 16)$

(c) sketch the curve. [3]

asympote at $x = 0$



Q8

8

The function f is defined by $f(x) = x^n - x$, $n \in \mathbb{N}, n \geq 2$. Determine the relationship between the value of n and the number of real solutions to the equation $f'(x) = 0$.

[4]

$f'(x) = nx^{n-1} - 1 = 0$

$x^{n-1} = \frac{1}{n}$

$\boxed{x = \frac{1}{\sqrt[n-1]{n}}}$

$n = 2$ $\quad x = \frac{1}{2} \longrightarrow$ 1 real solution

$n = 3$ $\quad x = \frac{1}{\sqrt{3}} \longrightarrow$ 2 real solutions

$n = 4$ $\quad x = \frac{1}{\sqrt[3]{4}} \longrightarrow$ 1 real solution

$n = 5$ $\quad x = \frac{1}{\sqrt[4]{5}} \longrightarrow$ 2 real solutions

If n is even, there is one real solution
If n is odd, there are two real solutions.

Q9

9

A curve is described by the equation $\frac{\sqrt{y}}{-1+\sqrt{x}} = \frac{1}{x}$, $x > 1$. Find $\frac{dy}{dx}$.

[3]

Make y the subject of the eqn.

$\sqrt{y} = \frac{-1+\sqrt{x}}{x}$

$y = \frac{(-1+\sqrt{x})^2}{x^2} = \frac{(-1+\sqrt{x})(-1+\sqrt{x})}{x^2}$

$= \frac{1 - 2\sqrt{x} + \sqrt{x}\sqrt{x}}{x^2}$

$= x^{-2} - 2x^{-3/2} + x^{-1}$

$\frac{dy}{dx} = -2x^{-3} - 2\left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1} + (-1)x^{-2}$

$= \boxed{-2x^{-3} + 3x^{-5/2} - x^{-2}}$

Q10

10

The curve with equation $y = ax^2 + bx + c$ passes through the point $(-1, 4)$. At the point $(2, 7)$ the gradient of the curve is 7. Find the values of a, b and c.

[5]

Sub in $(-1,4)$

$4 = a(-1)^2 + b(-1) + c$

$4 = a - b + c$ ①

Sub in $(2,7)$

$7 = a(2)^2 + b(2) + c$

$7 = 4a + 2b + c$ ②

$\frac{dy}{dx} = 2ax + b$

$7 = 2a(2) + b$

$b = 7 - 4a$ ③

① - ② to elimate c

$-3 = -3a - 3b$

$1 = a + b$

$b = 1 - a$

equate with b from ③

$b = 1 - a = 7 - 4a$

$3a = 6 \quad \therefore \boxed{a = 2}$

$\therefore b = 1 - (2) = -1 \quad \therefore \boxed{b = -1}$

sub a, b into ① $\therefore$ $4 = 2 - (-1) + c \quad \therefore \boxed{c = 1}$